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The Laplacian spectral radii of trees with degree sequences☆

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Abstract

In this paper, we characterize all extremal trees with the largest Laplacian spectral radius in the set of all trees with a given degree sequence. Consequently, we also obtain all extremal trees with the largest Laplacian spectral radius in the sets of all trees of order n with the largest degree, the leaves number and the matching number, respectively.

MSC: 05C50; 05C05; 15A48

Keywords: Laplacian spectral radius; Tree; Degree sequence; Majorization

1. Introduction

Let $G=(V,E)$ be a simple graph with vertex set $V(G)=\{v_1,\ldots,v_n\}$ and edge set $E(G)$. Denote by $d(v_i)$ the *degree* of vertex v_i. If $D(G)=\mathrm{diag}(d(u), u\in V)$ is the diagonal matrix of vertex degrees of G and $A(G)$ is the (0, 1) *adjacency matrix* of G, then the matrix $L(G)=D(G)-A(G)$ is called the *Laplacian matrix* of a graph G. It is obvious that $L(G)$ is positive semidefinite. Thus the spectral radius of $L(G)$ is equal to the largest eigenvalue of $L(G)$ and denoted by $\lambda(G)$. Moreover, $\lambda(G)$ is called the *Laplacian spectral radius* of G. The Laplacian matrices of graphs have received increasing attention in the past 20 years. The reader may be referred to [5,4,8,13] and the references therein. In particular, many researchers have investigated upper bounds for $\lambda(G)$ in terms of vertex degrees. Let us recall some known results.

In 1985, Anderson and Morley [1] showed that

$$\lambda(G)\leqslant \max\{d(u)+d(v) \mid (u,v)\in E(G)\}. \tag{1}$$

In 1997, Li and Zhang [7] gave a new upper bound.

$$\lambda(G)\leqslant 2+\sqrt{(r-2)(s-2)}, \tag{2}$$

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where $r=\max\{d(u)+d(v)|(u,v)\in E(G)\}$ and $s=\max\{d(u)+d(v)|(u,v)\in E(G)-(x,y)\}$ with $(x,y)\in E(G)$ such that $d(x)+d(y)=r$. In 2002, Shu et al. [10] gave an upper bound in terms of degree sequences. Assume that the degree sequence of G is $d_1\geqslant d_2\geqslant\cdots\geqslant d_n$. Then

$$\lambda(G)\leqslant d_n+\frac{1}{2}+\sqrt{\left(d_n-\frac{1}{2}\right)^2+\sum_{i=1}^{n}d_i(d_i-d_n)}. \tag{3}$$

In 2003, Stevanović [11] presented an upper bound for the spectral radius of a tree in terms of the largest vertex degree. He proved that if T be a tree with the largest vertex degree Δ, then

$$\lambda(T)<\Delta+2\sqrt{\Delta-1}. \tag{4}$$

In 2005, Rojo [9] improved Stevanović's result. He proved that if u be a vertex of T with the largest degree $d(u)=\Delta$ and denote by $k-1$ the largest distance from u to any other vertex of tree; for $j=1,\ldots,k-1$, let $\delta_j=\max\{d(v):\mathrm{dist}(v,u)=j\}$; then

$$\lambda(G)<\max\left\{\max_{2\leqslant j\leqslant k-2}\{\sqrt{\delta_j-1}+\delta_j+\sqrt{\delta_{j-1}-1}\},\sqrt{\delta_1-1}+\delta_1+\sqrt{\Delta},\Delta+\sqrt{\Delta}\right\}. \tag{5}$$

A nonincreasing sequence of nonnegative integers $\pi=(d_0,d_1,\ldots,d_{n-1})$ is called *graphic* if there exists a graph having π as its vertex degree sequence. Motivated by the recent results in terms of vertex degrees, we generally propose the following question.

Problem 1.1. *For a given graphic degree sequence* π, *let*

$$\mathscr{G}_\pi=\{G|G \text{ is connected with } \pi \text{ as its degree sequence}\}.$$

Find the upper (lower) bounds for the Laplacian spectral radius of all graphs G in $\mathscr{G}_\pi$ *and characterize all extremal graphs which attain the upper (lower) bounds.*

In other words, find all extremal graphs in $\mathscr{G}_\pi$ with largest Laplacian spectral radius. In this paper, we only consider a special case for the above problem, i.e., for a given degree sequence of some tree. The main result of this paper is as follows:

Theorem 1.2. *For a given degree sequence of some tree, let*

$$\mathscr{T}_\pi=\{T|T \text{ is tree with } \pi \text{ as its degree sequence}\}.$$

Then T^* *(see in Section 2) is a unique tree with largest Laplacian spectral radius in* $\mathscr{T}_\pi$.

The rest of the paper is organized as follows. In Section 2, some notations and preliminary results are presented. In Section 3, we present the proof of Theorem 1.2 and some corollaries.

2. Preliminary

For a given nonincreasing degree sequence $\pi=(d_0,d_1,\ldots,d_{n-1})$ of a tree with $n\geqslant 3$, we use breadth-first search method to define a special tree T^* with degree sequence π as follows. Assume that $d_m>1$ and $d_{m+1}=\cdots=d_{n-1}=1$ for $0\leqslant m<n-1$. Put $s_0=0$. Select a vertex v_{01} as a root and begin with v_{01} in layer 0. Put $s_1=d_0$ and select s_1 vertices $\{v_{11},\ldots,v_{1,s_1}\}$ in layer 1 such that they are adjacent to v_{01}. Thus $d(v_{01})=s_1=d_0$. We continue to construct all other layers by recursion. In general, put $s_t=d_{s_0+s_1+\cdots+s_{t-2}+1}+\cdots+d_{s_0+s_1+\cdots+s_{t-2}+s_{t-1}}-s_{t-1}$ for $t\geqslant 2$ and assume that all vertices in layer t have been constructed and are denoted by $\{v_{t1},\ldots,v_{ts_t}\}$ with $d(v_{t-1,1})=d_{s_0+\cdots+s_{t-2}+1},\ldots,d(v_{t-1,s_{t-1}})=d_{s_0+\cdots+s_{t-1}}$. Now using the induction hypothesis, we construct all vertices in layer $t+1$. Put $s_{t+1}=d_{s_0+\cdots+s_{t-1}+1}+\cdots+d_{s_0+\cdots+s_t}-s_t$. Select s_{t+1} vertices $\{v_{t+1,1},\ldots,v_{t+1,s_{t+1}}\}$ in layer $t+1$ such that $v_{t+1,i}$ is adjacent to v_{tr} for $r=1$ and $1\leqslant i\leqslant d_{s_0+\cdots+s_{t-1}+1}-1$ and for $2\leqslant r\leqslant s_t$ and $d_{s_0+\cdots+s_{t-1}+1}+d_{s_0+\cdots+s_{t-1}+2}+\cdots+d_{s_0+\cdots s_{t-1}+r-1}-r+2\leqslant i\leqslant d_{s_0+\cdots+s_{t-1}+1}+d_{s_0+\cdots+s_{t-1}+2}+\cdots+d_{s_0+\cdots+s_{t-1}+r}-r$. Thus $d(v_{tr})=d_{s_0+\cdots+s_{t-1}+r}$

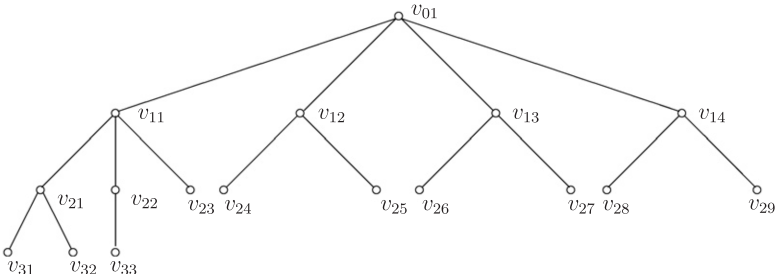

Fig. 1.

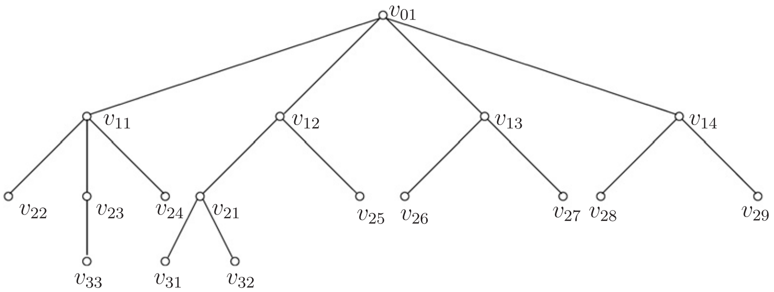

Fig. 2.

for $1 \leqslant r \leqslant s_t$. Assume that $m = s_0 + \cdots + s_{p-1} + q$. Put $s_{p+1} = d_{s_0+\cdots+s_{p-1}+1} + \cdots + d_{s_0+\cdots+s_{p-1}+q} - q$ and select s_{p+1} vertices $\{v_{p+1,1}, \ldots, v_{p+1,s_{p+1}}\}$ in layer $p+1$ such that $v_{p+1,i}$ is adjacent to v_{pr} for $1 \leqslant r \leqslant q$ and $d_{s_0+\cdots+s_{t-1}+1} + \cdots + d_{s_0+\cdots+s_{p-1}+2} + \cdots + d_{s_0+\cdots s_{p-1}+r-1} - r + 2 \leqslant i \leqslant d_{s_0+\cdots+s_{p-1}+1} + \cdots + d_{s_0+\cdots+s_{p-1}+2} + \cdots + d_{s_0+\cdots+s_{p-1}+r} - r$. Thus $d(v_{p,i}) = d_{s_0+\cdots+s_{p-1}+i}$ for $1 \leqslant i \leqslant q$. In this way, we obtain a tree T^*. It is easy to see that T^* is of order n with degree sequence π.

For example, for a given degree sequence $\pi = (4, 4, 3, 3, 3, 3, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$, T^* is the tree of order 17 (see Fig. 1). There is a vertex v_{01} in layer 0; four vertices $v_{11}, v_{12}, v_{13}, v_{14}$ in layer 1; nine vertices $v_{21}, v_{22}, \ldots, v_{29}$ in layer 2; three vertices v_{31}, v_{32}, v_{33} in layer 3. Moreover, $s_0 = 0$, $s_1 = d_0 = 4$, $s_2 = d_1 + d_2 + d_3 + d_4 - s_1 = 4 + 3 + 3 + 3 - 4 = 9$, $s_3 = d_5 + \cdots + d_{13} - s_2 = 3$, $m = s_1 + q = 4 + 2 = 6$.

For a graph with a root v_0, we call the distance the *height* $h(v) = \mathrm{dist}(v, v_0)$ of a vertex v.

Definition 2.1. Let $T = (V, E)$ be a tree with root v_0. A well-ordering $\prec$ of the vertices is called breadth-first search ordering with nonincreasing degrees (BFS-ordering for short) if the following holds for all vertices $u, v \in V$:

(1) $u \prec v$ implies $h(u) \leqslant h(v)$;
(2) $u \prec v$ implies $d(u) \geqslant d(v)$;
(3) if there are two edges $uu_1 \in E(T)$ and $vv_1 \in E(T)$ such that $u \prec v$, $h(u) = h(u_1) + 1$ and $h(v) = h(v_1) + 1$, then $u_1 \prec v_1$.

We call trees that have a BFS-ordering of its vertices a BFS-tree.

All trees have an ordering which satisfy the conditions (1) and (3) by using breadth-first search, but not all tree have a BFS-ordering. For example, the following tree T of order 17 has not a BFS-ordering with degree sequence $\pi = (4, 4, 3, 3, 3, 3, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$ (see Fig. 2).

In fact, it is easy to show the following assertion holds.

Proposition 2.2. *For a given degree sequence π of some tree, there exists a unique tree T^* with degree sequence π having a BFS-ordering. Moreover, any two trees with the same degree sequences and having BFS-ordering are isomorphic.*

We recall the notion of majorization. Let $\pi = (d_0, \ldots, d_{n-1})$ and $\pi' = (d'_0, \ldots, d'_{n-1})$ be two nonincreasing sequences. If $\sum_{i=0}^{k} d_i \leqslant \sum_{i=0}^{k} d'_i$ and $\sum_{i=0}^{n-1} d_i = \sum_{i=0}^{n-1} d'_i$, then the sequence π' is said to *major* the sequence π and denoted by $\pi \lhd \pi'$. It is known that the following result holds (see [3]).

Proposition 2.3 (*Erdös and Gallai [3]*). *Let $\pi = (d_0, \ldots, d_{n-1})$ and $\pi' = (d'_0, \ldots, d'_{n-1})$ be two nonincreasing graphic degree sequences. If $\pi \lhd \pi'$, then there exists a series graphic degree sequences $\pi_1, \ldots, \pi_k$ such that $\pi \lhd \pi_1 \lhd \cdots \lhd \pi_k \lhd \pi'$, and only two components of π_i and π_{i+1} are different from* 1.

We also need the following Lemma from [8]

Lemma 2.4 (*Merris [8]*). *Let G be a simple graph with the degree diagonal matrix $D(G)$ and the adjacency matrix $A(G)$. Denote by $\lambda(G)$ and $\mu(G)$ the spectral radii of the matrices $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$, respectively. Then $\lambda(G) = \mu(G)$ if and only if G is a bipartite graph.*

By the Perron–Frobenius Theorem, if G is connected, $\mu(G)$ is the largest eigenvalue of $Q(G)$ and simple. Moreover, there exists a unique positive unit eigenvector corresponding to $\mu(G)$. We refer to such an eigenvector as the *Perron vector* of G.

3. Main results

In order to prove Theorem 1.2, we need some lemmas:

Lemma 3.1. *Let $G = (V(G), E(G))$ be a connected simple graph with $uv_i \in E(G)$ and $wv_i \notin E(G)$ for $i = 1, \ldots, k$. Let $G' = (V(G'), E(G'))$ be a new graph from G by deleting edges uv_i and adding edges wv_i for $i = 1, \ldots, k$. Let f be a Perron vector of $Q(G)$. If $f(w) \geqslant f(u)$, then $\mu(G) < \mu(G')$.*

Proof. Let $\mathbf{S}$ be the set of all unit vectors in $\mathbf{R}^n$. Then by the Rayleigh quotient of $Q(G)$ on vectors g on V and [8],

$$\mu(G) = \max_{g \in \mathbf{S}} g^{\mathrm{T}} Q(G) g = \max_{g \in \mathbf{S}} \sum_{xy \in E(G)} (g(x) + g(y))^2 = \sum_{xy \in E(G)} (f(x) + f(y))^2.$$

Moreover

$$\sum_{xy \in E(G')} (f(x) + f(y))^2 - \sum_{xy \in E(G)} (f(x) + f(y))^2$$
$$= 2(f(w) - f(u)) \sum_{i=1}^{k} (f(w) + f(u) + 2f(v_i)) \geqslant 0.$$

Hence

$$\mu(G') = \max_{g \in \mathbf{S}} \sum_{xy \in E(G')} (g(x) + g(y))^2 \geqslant \sum_{xy \in E(G')} (f(x) + f(y))^2 \geqslant \mu(G).$$

If $\mu(G') = \mu(G)$, then f is the Perron vector of G'. Hence $Q(G) f = \mu(G) f$ and $Q(G') f = \mu(G') f$. From the equation corresponding to vertex u, we have

$$\mu(G) f(u) = d_G(u) f(u) + \sum_{i=1}^{k} f(v_i) + \sum_{vu \in E(G), v \neq v_i} f(v)$$

and

$$\mu(G') f(u) = d_{G'}(u) f(u) + \sum_{vu \in E(G')} f(v).$$

Therefore $f(v_i) = 0$ by $d_G(u) > d_{G'}(u)$. It is impossible. Hence $\mu(G') > \mu(G)$. $\square$

Corollary 3.2. *Let G be a connected simple graph with degree sequence π and $d_G(u) - d_G(w) = k > 0$. Let f be a Perron vector of $Q(G)$. If $f(w) \geqslant f(u)$, then there exists a connected simple graph G' with degree sequence π such that $\mu(G) < \mu(G')$.*

Proof. Since $d_G(u) - d_G(w) = k$, there exist k vertices such that $uv_i \in E(G)$ and $wv_i \notin E(G)$ for $i = 1, \dots, k$. It follows from Lemma 3.1 that the assertion holds. □

Lemma 3.3. *Let $G = (V(G), E(G))$ be a connected simple graph. Assume that $v_1u_1 \in E(G)$, $v_2u_2 \in E(G)$, $v_1v_2 \notin E(G)$ and $u_1u_2 \notin E(G)$. Let $G' = (V(G'), E(G'))$ be a new graph from G by deleting edges v_1u_1 and v_2u_2 and adding edges v_1v_2 and u_1u_2. Let f be the Perron vector of G. If $f(v_1) \geqslant f(u_2)$ and $f(v_2) \geqslant f(u_1)$, then $\mu(G') \geqslant \mu(G)$. Moreover, if one of the two inequalities is strict, then $\mu(G') > \mu(G)$.*

Proof. Since f is the Perron vector of G,

$$\mu(G) = \max_{g \in \mathbf{S}} g^{\mathrm{T}} Q(G) g = \max_{g \in \mathbf{S}} \sum_{xy \in E(G)} (g(x) + g(y))^2 = \sum_{xy \in E(G)} (f(x) + f(y))^2.$$

On the other hand, by $f(v_1) \geqslant f(u_2)$ and $f(v_2) \geqslant f(u_1)$, we have

$$\sum_{xy \in E(G')} (f(x) + f(y))^2 - \sum_{xy \in E(G)} (f(x) + f(y))^2 = 2(f(v_2) - f(u_1))(f(v_1) - f(u_2)) \geqslant 0.$$

Hence

$$\mu(G') = \max_{g \in \mathbf{S}} \sum_{xy \in E(G')} (g(x) + g(y))^2 \geqslant \sum_{xy \in E(G')} (f(x) + f(y))^2 \geqslant \mu(G).$$

Moreover, if $\mu(G') = \mu(G)$, then f is also the Perron vector of G'. From the equation corresponding to vertex v_1 in $Q(G)f = \mu(G)f$ and $Q(G')f = \mu(G')f$, we have

$$\mu(G) f(v_1) = d_G(v_1) f(v_1) + f(u_1) + \sum_{wv_1 \in E(G) \cap E(G')} f(w)$$

and

$$\mu(G') f(v_1) = d_{G'}(v_1) f(v_1) + f(v_2) + \sum_{wv_1 \in E(G) \cap E(G')} f(w).$$

Hence $f(u_1) = f(v_2)$ by $d_G(v_1) = d_{G'}(v_1)$. Similarly, we show that $f(v_1) = f(u_2)$. □

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Assume that T is a tree in $\mathscr{T}_\pi$ with largest Laplacian spectral radius, where $\pi = (d_0, \dots, d_{n-1})$ and $d_0 \geqslant d_1 \geqslant \cdots \geqslant d_{n-1}$. By Lemma 2.4, $\lambda(T) = \mu(T)$. Let f be the Perron vector of T. Without loss of generality, we may assume that $V(T) = \{v_0, \dots, v_{n-1}\}$ such that $f(v_i) \geqslant f(v_j)$ for $i < j$, i.e., they are denoted with respect to $f(v)$ in nonincreasing order. Put $V_i = \{v : \mathrm{dist}(v, v_0) = i\}$ for $i = 0, \dots, p+1$ such that $V(T) = \bigcup_{i=0}^{p+1} V_i$. Denote by $|V_i| = s_i$ for $i = 1, \dots, p+1$. We now may relabel the vertices of $V(T)$ by the recursion method. For V_0, relabel v_0 by v_{01} and as the root of tree T. For all vertices of V_1, which consists of all neighbors of vertices in V_0, may be relabeled

$$v_{11}, \dots, v_{1,s_1}$$

and satisfy the following conditions:

$$f(v_{11}) \geqslant f(v_{12}) \geqslant \cdots \geqslant f(v_{1,s_1})$$

and

$$f(v_{1i}) = f(v_{1j}) \quad \text{implies } d(v_{1i}) \geqslant d(v_{1j}) \text{ for } 1 \leqslant i < j \leqslant s_1.$$

Moreover, $s_1 = d(v_{01})$. Generally, we assume that all vertices of V_i are relabeled $\{v_{i1}, \ldots, v_{i,s_i}\}$ for $i = 1, \ldots, t$. Now consider all vertices in V_{t+1}. Since T is tree, it is easy to see that

$$s_{t+1} = |V_{t+1}| = d(v_{t1}) + \cdots + d(v_{t,s_t}) - s_t.$$

Hence for $1 \leqslant r \leqslant s_t$, all neighbors in V_{t+1} of v_{tr} are relabeled

$$v_{t+1,d(v_{t1})+\cdots+d(v_{t,r-1})-(r-1)+1}, \ldots, v_{t+1,d(v_{t1})+\cdots+d(v_{t,r})-r}$$

and satisfy the conditions:

$$f(v_{t+1,i}) \geqslant f(v_{t+1,j}) \tag{6}$$

and

$$f(v_{t+1,i}) = f(v_{t+1,j}) \quad \text{implies } d(v_{t+1,i}) \geqslant d(v_{t+1,j}) \tag{7}$$

for $d(v_{t1}) + \cdots + d(v_{t,r-1}) - (r-1) + 1 \leqslant i < j \leqslant d(v_{t1}) + \cdots + d(v_{t,r}) - r$. In this way, we have relabeled all vertices of $V(T) = \bigcup_{i=0}^{p+1} V_i$. Therefore, we are able to define a well ordering of vertices in $V(T)$ as follows:

$$v_{ik} \prec v_{jl} \ \text{ if } 0 \leqslant i < j \leqslant p+1 \ \text{ or } \ i = j \ \text{ and } \ 1 \leqslant k < l \leqslant s_i. \tag{8}$$

We need to show that this well ordering is a BFS-ordering of tree T. In other words, T is isomorphic to T^*.

In order to show this assertion, we first prove that the following two equations hold.

$$f(v_{h1}) \geqslant f(v_{h2}) \geqslant \cdots \geqslant f(v_{h,s_h}) \geqslant f(v_{h+1,1}) \tag{9}$$

and

$$d(v_{h1}) \geqslant d(v_{h2}) \geqslant \cdots \geqslant d(v_{h,s_h}) \geqslant d(v_{h+1,1}) \tag{10}$$

for $h = 0, \ldots, p+1$ by the induction on h.

For $h = 0$, clearly, (9) and (10) hold. Assume that (9) and (10) hold for $h = t$. We consider $h = t+1$. Suppose that $f(v_{t+1,i}) < f(v_{t+1,j})$ for $1 \leqslant i < j \leqslant s_{t+1}$. Then there exist two vertices v_{tk} and v_{tl} with $k < l$ in layer t such that $v_{tk}v_{t+1,i} \in E(T)$ and $v_{tl}v_{t+1,j} \in E(T)$. Then $f(v_{tk}) \geqslant f(v_{tl})$. Let T' be a graph from T by adding the edges $v_{tk}v_{t+1,j}$ and $v_{tl}v_{t+1,i}$ and deleting the edges $v_{tk}v_{t+1,i}$ and $v_{tl}v_{t+1,j}$. By Lemma 3.3, T' is a tree with the same degree sequence π and $\mu(T') > \mu(T)$. It contradicts to T being the largest Laplacian spectral radius in $\mathscr{T}_\pi$. Similarly, we also show that $f(v_{t+1,s_{t+1}}) \geqslant f(v_{t+2,1})$. Hence (9) holds. Suppose that $d(v_{t+1,i}) < d(v_{t+1,j})$ for $1 \leqslant i < j \leqslant s_{t+1}$. Then $f(v_{t+1,i}) \geqslant f(v_{t+1,j})$ and let $\delta = d(v_{t+1,j}) - d(v_{t+1,j}) > 0$. By Corollary 3.2, T is not the largest Laplacian spectral radius in $\mathscr{T}_\pi$. Hence (10) holds also.

Therefore, we have

$$f(v_{01}) \geqslant f(v_{11}) \geqslant \cdots \geqslant f(v_{1,s_1}) \geqslant f(v_{21}) \geqslant \cdots \geqslant f(v_{2,s_2}) \geqslant \cdots \geqslant f(v_{p+1,s_{p+1}}) \tag{11}$$

and

$$\begin{aligned} &d(v_{01}) = d_0, d(v_{11}) = d_1, \ldots, d(v_{1,s_1}) = d_{s_1}, \\ &d(v_{21}) = d_{s_1+1}, \ldots, d(v_{2,s_2}) = d_{s_1+s_2}, \ldots, \\ &d(v_{p+1,1}) = d_{s_1+\cdots+s_p+1}, \ldots, d(v_{p+1,s_{p+1}}) = d_{n-1}. \end{aligned} \tag{12}$$

By (8), (11) and (12), it is easy to see that this well ordering satisfies the conditions (1)–(3) in Definition 2.1. Hence T has a BFS-ordering. Further, by Proposition 2.2, T is isomorphic to T^*. So T^* is a unique tree in $\mathscr{T}_\pi$ having the largest Laplacian spectral radius by Lemma 2.4. $\square$

From the proof of Theorem 1.2, it is easy to see that we have the following:

Corollary 3.4. *For a given tree degree sequence π, a tree T has the largest Laplacian spectral radius in $\mathscr{T}_\pi$ if and only if T has a BFS-ordering. Moreover, the BFS-ordering is consistent with the Perron vector f of T in such a way that $f(u) > f(v)$ implies $u \prec v$.*

Theorem 3.5. *Let π and π' be two different tree degree sequences with the same order. Let T^* and $(T')^*$ have the largest Laplacian spectral radii in $\mathscr{T}_\pi$ and $\mathscr{T}_{\pi'}$, respectively. If $\pi \lhd \pi'$, then $\lambda(T^*) < \lambda((T')^*)$.*

Proof. By Proposition 2.3, without loss of generality, we may assume that $\pi = (d_0, \ldots, d_{n-1})$ and $\pi' = (d'_0, \ldots, d'_{n-1})$ with $d_i = d'_i$ for $i \neq p, q$, and $d_p = d'_p - 1$, $d_q = d'_q + 1$, $0 \leqslant p < q \leqslant n-1$. Moreover, let π and π' be degree sequences of T^* and $(T')^*$, respectively. By Corollary 3.4, the BFS-ordering of T^* is consistent with the Perron vector f of T^* in such a way that $f(u) > f(v)$ implies $u \prec v$. Hence we may assume that the vertices of T^* are ordered $\{v_0, \ldots, v_{n-1}\}$ such that $d(v_i) = d_i$ for $i = 0, \ldots, n-1$ and $f(v_0) \geqslant f(v_1) \geqslant \cdots \geqslant d(v_{n-1})$. Moreover, since $d_q = d'_q + 1 \geqslant 2$, there exists a vertex v_k with $k > q$ such that $v_k v_q \in E(T^*)$ and $v_k v_p \notin E(T^*)$. Let T_1 be a tree from T^* by adding the edge $v_k v_p$ and deleting $v_k v_q$. Then by Lemma 3.1, $\mu(T^*) < \mu(T_1)$. Moreover, the degree sequence of T_1 is π'. Hence $\mu(T_1) \leqslant \mu((T')^*)$ with equality if and only if T_1 is $(T')^*$. Hence by Lemma 2.4, $\lambda(T^*) < \lambda((T')^*)$. $\square$

From Theorems 1.2 and 3.5, we may deduce extremal graphs with the largest Laplacian spectral radius in some class of graphs. For example, let $\mathscr{T}^{(1)}_{n,s}$ be the set of all trees of order n with s leaves, $\mathscr{T}^{(2)}_{n,\Delta}$ be the set of all trees of order n with the largest degree Δ, $\mathscr{T}^{(3)}_{n,\alpha}$ be the set of all trees of order n with the independence number α and $\mathscr{T}^{(4)}_{n,\beta}$ be the set of all trees of order n with the matching number β.

Corollary 3.6. *A tree T_1 has the largest Laplacian spectral radius in $\mathscr{T}^{(1)}_{n,s}$ if and only if T_1 is a star with paths of almost the same length to each of its s leaves* (*in other words, let $n-1 = sq + t$, $0 \leqslant t < s$ and T^* is obtained from t paths of order $q+2$ and $s-t$ paths of order $q+1$ by identifying one end of the s paths.*).

Proof. Let T be any tree in $\mathscr{T}^{(1)}_{n,s}$ with the nonincreasing degree sequence $\pi = (d_0, \ldots, d_{n-1})$. Thus $d_{n-s-1} > 1$ and $d_{n-s} = \cdots = d_{n-1} = 1$. Let T^* have a BFS ordering tree with the degree sequence $\pi^* = (s, 2, \ldots, 2, 1, \ldots, 1)$, where there are the number s of 1 in π^*. By Corollary 2.2, T^* a star with paths of almost the same length to each of its s leaves. Moreover, it is easy to see that $\pi \lhd \pi^*$. By Theorem 3.5, the assertion holds. $\square$

Corollary 3.7. *A tree T_2 has the largest Laplacian spectral radius in $\mathscr{T}^{(2)}_{n,\Delta}$ with $\Delta \geqslant 3$ if and only if T_2 is T^* in Theorem* 1.2 *with degree sequence π^* which is as follows*: *Denote $p = \lceil \log_{(\Delta-1)} \frac{n(\Delta-2)+2}{\Delta} \rceil - 1$ and $n - \frac{\Delta(\Delta-1)^p - 2}{\Delta-2} = (\Delta-1)r + q$ for $0 \leqslant q < \Delta - 1$. If $q = 0$, put $\pi^* = (\Delta, \ldots, \Delta, 1, \ldots, 1)$ with the number $\frac{\Delta(\Delta-1)^{p-1}-2}{\Delta-2} + r$ of degree Δ. If $1 \leqslant q$, put $\pi^* = (\Delta, \ldots, \Delta, q, 1, \ldots, 1)$ with the number $\frac{\Delta(\Delta-1)^{p-1}-2}{\Delta-2} + r$ of degree Δ.*

Proof. For any tree T of order n with the largest degree Δ, let $\pi = (d_0, \ldots, d_{n-1})$ be the nonincreasing degree sequence of T. Assume that T^* has $p+2$ layers. Then there is a vertex in layer 0 (i.e., root), there are Δ vertices in layer 1, there are $\Delta(\Delta-1)$ vertices in layer $2, \ldots,$ there are $\Delta(\Delta-1)^{p-1}$ vertices in layer p, there are at most $\Delta(\Delta-1)^p$ vertices in layer $p+1$. Hence

$$1 + \Delta + \Delta(\Delta-1) + \cdots + \Delta(\Delta-1)^{p-1} < n \leqslant 1 + \Delta + \Delta(\Delta-1) + \cdots + \Delta(\Delta-1)^p.$$

Thus

$$\frac{\Delta(\Delta-1)^p - 2}{\Delta - 2} < n \leqslant \frac{\Delta(\Delta-1)^{p+1} - 2}{\Delta - 2}.$$

Hence

$$p = \left\lceil \log_{(\Delta-1)} \frac{n(\Delta-2)+2}{\Delta} \right\rceil - 1$$

and there exist integers r and $0 \leqslant q < \Delta - 1$ such that

$$n - \frac{\Delta(\Delta-1)^p - 2}{\Delta - 2} = (\Delta-1)r + q.$$

Therefore degrees of all vertices from layer 0 to layer $p-1$ are Δ and there are r vertices in layer p with degree Δ. Denote by $m = \Delta(\Delta-1)^{p-1} - 2/\Delta - 2 + r - 1$. Then there are $m+1$ vertices with degree Δ in T^*. Hence the degree sequence of $T^* \in \mathscr{T}_{n,\Delta}$ is $\pi^* = (d_0^*, \ldots, d_{n-1}^*)$ with $d_0^* = \cdots = d_m^* = \Delta$, $d_{m+1}^* = \cdots = d_{n-1}^* = 1$ for $q = 0$; and is $\pi^* = (d_0^*, \ldots, d_{n-1}^*)$ with $d_0^* = \cdots = d_m^* = \Delta$, $d_{m+1}^* = q$, $d_{m+2}^* = \cdots = d_{n-1}^* = 1$. It follows from $d_0 \leqslant \Delta$ that $\sum_{i=0}^k d_i \leqslant \sum_{i=0}^k d_i^*$ for $k = 0, \ldots, m$. Further by $d_i^* = 1 \leqslant d_i$ for $k = m+2, \ldots, n-1$, we have

$$\sum_{i=0}^{k} d_i = 2(n-1) - \sum_{i=k+1}^{n-1} d_i \leqslant 2(n-1) - \sum_{i=k+1}^{n-1} d_i^* = \sum_{i=0}^{k} d_i^*$$

for $k = m+1, \ldots, n-1$. Thus $\pi \lhd \pi^*$. Hence by Theorems 1.2 and 3.5, $\lambda(T) \leqslant \lambda(T^*)$ with equality if and only if $T = T^*$. $\square$

Corollary 3.8 (*Zhang [12]*). *Let $\mathscr{T}_{n,\alpha}^{(3)}$ be the set of all trees of order n with the independence number α. A tree T_3 has the largest Laplacian spectral radius in $\mathscr{T}_{n,\alpha}^{(3)}$ if and only if T_3 is T^* in Theorem* 1.2 *with degree sequence $\pi^* = (\alpha, 2, \ldots, 2, 1, \ldots, 1)$ the numbers $n - \alpha - 1$ of 2 and α of 1.*

Proof. For any tree T of order n with the independence number α, let I be an independent set of T with the independence number α and $\pi = (d_0, \ldots, d_{n-1})$ be the nonincreasing degree sequence of T. If there exists a pendent vertex u of degree 1 with $u \notin I$, then there exists a vertex $v \in I$ with $(u, v) \in E(T)$. Hence $I \cup \{u\} \backslash \{v\}$ is an independent set of T with the size α. Therefore, there exists an independent set of T with α which contains all pendent vertices of T. Hence there are at most α pendent vertices. Then $d_{n-\alpha-1} \geqslant 2$ and $\pi \lhd \pi^*$. Therefore by Theorems 1.2 and 3.5, the assertion holds. $\square$

Corollary 3.9 (*Guo [6]*). *Let $\mathscr{T}_{n,\beta}^{(4)}$ be the set of all trees of order n with the matching number β. A tree T_4 has the largest Laplacian spectral radius in $\mathscr{T}_{n,\beta}^{(4)}$ if and only if T_4 is T^* in Theorem* 1.2 *with degree sequence $\pi^* = (n - \beta, 2, \ldots, 2, 1, \ldots, 1)$ and the number $n - \beta$ of 1.*

Proof. For any tree T of order n with the matching number β, let $\pi = (d_0, \ldots, d_{n-1})$ be the nonincreasing degree sequence of T. Let M be a matching of T with the matching number β. Since T is connected, there are at least β vertices in T such that their degrees are at least two. Hence $d_{\beta-1} \geqslant 2$. Then $\pi \lhd \pi^*$. Therefore by Theorems 1.2 and 3.5, the assertion holds. $\square$

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